JOURNAL OF APPROXIMATION THEORY 4, 288–294 (1971)

Cauchy's Problem for Almost Linear Elliptic Equations in Two Independent Variables, II

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Communicated by Yudell L. Luke

Received January 27, 1970

1. INTRODUCTION

Due to the unstable dependence of the solution on the initial data, Cauchy's problem for elliptic equations is well known to be improperly posed in the sense of Hadamard (cf. [2, p. 108]). Such problems arise, however, in the study of free boundary problems (cf. [2, p. 622]), in mathematical physics and hence attention has been focused on methods of solution that are suitable for analytic approximation and numerical computation. We note that the Cauchy–Kowalewski theorem is no more suitable here than it is for hyperbolic equations. For second-order equations in two independent variables, the approximation problem is in satisfactory condition [1; 2, p. 623–631; 4]. However, the situation for higher order equations or equations in more than two independent variables is not so well off from a computational viewpoint, since the only available method is to convert an elliptic problem in only $n + 1$ variables into a hyperbolic problem in no less than $2n + 1$ variables [2, p. 614-621]. In this note, we show how previous results obtained by the author in [1] for second-order almost linear (or semilinear) equations can be adapted to give approximation techniques for a quite general class of higher order equations.

In [1], the equation

$$\Delta u = f(x, y, u, u_x, u_y) \tag{1.1}$$

was considered, with Cauchy data prescribed on a given analytic arc L. Without loss of generality, we assumed L was the x axis. In *conjugate coordinates* [3, 5]

$$z = x + iy,$$

and (1.2)

$$z^* = x - iy,$$

Eq. (1.1) became

$$U_{zz^*} = F(z, z^*, U, U_z, U_{z^*}), \tag{1.3}$$

with initial data prescribed on the plane $z = z^*$. Under the assumption that $F(z, z^*, \xi_1, \xi_2, \xi_3)$ was an analytic function of its five variables, it was shown in [1] that $S(z, z^*) = U_{zz^*}$ is the (unique) fixed point of a contraction mapping in an appropriate Banach space of analytic functions, and that U could be easily obtained from S by integration and a knowledge of the Cauchy data. We now show how the Cauchy problem

$$\begin{gathered} \Delta^n u = f\left(x, y, u, u_x, u_y, \ldots, \frac{\partial^{l+m}\Delta^j u}{\partial x^l\, \partial y^m}, \ldots, \frac{\partial \Delta^{n-1} u}{\partial x}, \frac{\partial \Delta^{n-1} u}{\partial y}\right), \\ l = 0, 1, \ldots, n; \qquad m = 0, 1, \ldots, n; \qquad l + m + 2j \leqslant 2n - 1, \end{gathered} \tag{1.4}$$

$$u(x, 0) = \varphi_0(x), \quad \frac{\partial^k u(x, 0)}{\partial y^k} = \varphi_k(x); \qquad k = 1, 2, \ldots, 2n - 1, \tag{1.5}$$

can be reduced to a Cauchy problem for

$$\Delta u = f(x, y, A_1(u), \ldots, A_N(u)), \tag{1.6}$$

where A_i, $i = 1, 2, \ldots, N$, are operators satisfying a certain type of Lipschitz condition in an appropriate Banach space. This latter problem will then be solved using techniques similar to those used in solving Cauchy's problem for Eq. (1.1). Note that again there is no loss of generality in assuming that the Cauchy data is prescribed along the x axis.

II. Reduction and Solution of Higher Order Cauchy Problems

In complex form, the Cauchy problem (1.4), (1.5) becomes

$$\frac{\partial^{2n} U}{\partial z^n\, \partial z^{*n}} = F\left(z, z^*, U, \ldots, \frac{\partial^{p+q} U}{\partial z^p\, \partial z^{*q}}, \ldots, \frac{\partial^{2n-1} U}{\partial z^{n-1}\, \partial z^{*n}}, \frac{\partial^{2n-1} U}{\partial z^n\, \partial z^{*n-1}}\right), \tag{2.1}$$

$$\begin{gathered} U(z, z^*) = \varphi_0(z); \qquad z = z^*, \\ i^k \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right)^k U(z, z^*) = \varphi_k(z); \qquad z = z^*, \quad k = 1, \ldots, 2n - 1, \end{gathered} \tag{2.2}$$

where $U(z, z^*) = u((z + z^*/2, (z - z^*)/2i)$, $p = 0, 1, \ldots, n$; $q = 0, 1, \ldots, n$; $p + q \leqslant 2n - 1$. We assume that, as a function of its first two arguments, $F(z, z^*, \xi_1, \ldots, \xi_N)$ is holomorphic in a bicylinder $\mathfrak{S} \times \mathfrak{S}^*$, where $\mathfrak{S}^* = \{z \mid z^* \in \mathfrak{S}\}$, and as a function of its last N variables, it is holomorphic in a sufficiently large ball about the origin. We further assume that $\mathfrak{S}$ is simply connected, contains the origin, is symmetric with respect to conjugation, i.e., $\mathfrak{S} = \mathfrak{S}^*$, and that $\varphi_k(z)$, $k = 0, 1, \ldots, 2n - 1$, are holomorphic in $\mathfrak{S}$.

We would like to emphasize that it is necessary for us to restrict ourselves to equations of the form (1.4) in order that there do not appear any terms of the form $\partial^{n+1}U/\partial z^{n+1}$, $\partial^{n+1}U/\partial z^{*n+1}$, $\partial^{2n-1}U/\partial z^{n-2}\,\partial z^{*n+1}$ etc., when Eq. (1.4) is written in terms of conjugate coordinates. For example, our analysis is not applicable to equations such as

$$\Delta^n u = f\left(x, y, u, \frac{\partial u}{\partial x}, \dots, \frac{\partial^{2n-1}u}{\partial x^{2n-1}}\right). \tag{2.3}$$

We note that the same type of restriction was also encountered by I. N. Vekua [5, p. 174-228] in his study of the analytic theory of higher order linear elliptic equations in two independent variables.

We now proceed with the reduction of the Cauchy problem (1.4), (1.5) to the second-order operator Eq. (1.6). Let

$$U^{(1)} = \frac{\partial^2 U}{\partial z\,\partial z^*} = \frac{1}{4}\Delta u. \tag{2.4}$$

Then

$$\begin{aligned} U_z &= \int_z^{z^*} U^{(1)}(z, \xi^*)\,d\xi^* + U_z(z, z) \\ &= \int_z^{z^*} U^{(1)}(z, \xi^*)\,d\xi^* + \tfrac{1}{2}[\varphi_0'(z) - i\varphi_1(z)], \end{aligned} \tag{2.5}$$

$$\begin{aligned} U_{z^*} &= \int_{z^*}^{z} U^{(1)}(\xi, z^*)\,d\xi + U_{z^*}(z, z) \\ &= \int_{z^*}^{z} U^{(1)}(\xi, z^*)\,d\xi + \tfrac{1}{2}[\varphi_0'(z) + i\varphi_1(z)], \end{aligned} \tag{2.6}$$

$$\begin{aligned} U &= \int_{z^*}^{z} U_\xi(\xi, z^*)\,d\xi + U(z, z) \\ &= \int_{z^*}^{z} \left\{\int_\xi^{z^*} U^{(1)}(\xi, \xi^*)\,d\xi^* + \tfrac{1}{2}[\varphi_0'(\xi) - i\varphi_1(\xi)]\right\} d\xi + \varphi_0(z). \end{aligned} \tag{2.7}$$

By using Eqs. (2.4)–(2.7), $\partial^{p+q}U/\partial z^p\,\partial z^{*q}$, for $p = 0, 1,\dots, n$; $q = 0, 1,\dots, n$; $p + q \leqslant 2n - 1$, can all be computed in terms of a linear combination of $\partial^{p+q}U^{(1)}/\partial z^p\,\partial z^{*q}$ and its integrals, $p = 0, 1,\dots, n-1$; $q = 0, 1,\dots, n-1$; $p + q \leqslant 2n - 3$. Furthermore, Eqs. (2.4) and (1.5) allow $\partial^k u^{(1)}(x, 0)/\partial y^k$, $k = 1, 2,\dots, 2n-3$, to be computed in terms of the Cauchy data for u. Hence we are led to the following Cauchy problem for $U^{(1)}$:

$$\frac{\partial^{2n-2}U^{(1)}}{\partial z^{n-1}\,\partial z^{*n-1}} = F(z, z^*, A_1^{(1)}(U^{(1)}),\dots, A_N^{(1)}(U^{(1)})), \tag{2.8}$$

$$\begin{aligned} U^{(1)}(z, z^*) &= \varphi_0^{(1)}(z); \qquad z = z^*, \\ i^k\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*}\right)^k U^{(1)}(z, z^*) &= \varphi_k^{(1)}(z); \qquad z = z^*, \quad k = 1, 2,\dots, 2n-3, \end{aligned} \tag{2.9}$$

where $A_i^{(1)}$, $i = 1, 2,..., N$, are integral operators on HB into HB, $HB \equiv HB(\Delta\rho, \Delta\rho^*)$ being the Banach space of functions of two complex variables which are holomorphic and bounded in

$$\Delta\rho \times \Delta\rho^*, \qquad \Delta\rho = \{z \mid |z| < \rho\}, \qquad \Delta\rho^* = \{z \mid z^* \in \Delta\rho\},$$

with norm

$$\| S \| = \sup_{\Delta\rho\times\Delta\rho^*} | S(z, z^*)|. \tag{2.10}$$

More precisely, $A_1^{(1)}$ is defined by Eq. (2.7), $A_2^{(1)}$ by Eq. (2.6), $A_3^{(1)}$ by Eq. (2.5), and $A_i^{(1)}$, $i > 3$, is obtained by repeated differentiation of Eq. (2.5) or (2.6). It is easily seen that each $A_i^{(1)}$, $i = 1,..., N$, satisfies the condition

$$\begin{aligned} &\| A_i^{(1)}(U_1^{(1)}) - A_i^{(1)}(U_2^{(1)})\| \\ &\leqslant M_i^{(1)} \Big\{\| U_1^{(1)} \quad U_2^{(1)} \| + \cdots + \left\| \frac{\partial^{p+q}U_1^{(1)}}{\partial z^p\, \partial z^{*q}} - \frac{\partial^{p+q}U_2^{(1)}}{\partial z^p\, \partial z^{*q}} \right\| + \cdots \\ &\quad + \left\| \frac{\partial^{2n-3}U_1^{(1)}}{\partial z^{n-2}\, \partial z^{*n-1}} - \frac{\partial^{2n-3}U_2^{(1)}}{\partial z^{n-2}\, \partial z^{*n-1}} \right\| + \left\| \frac{\partial^{2n-3}U_1^{(1)}}{\partial z^{n-1}\, \partial z^{*n-2}} - \frac{\partial^{2n-3}U_2^{(1)}}{\partial z^{n-1}\, \partial z^{*n-2}} \right\| \Big\} \end{aligned} \tag{2.11}$$

for some positive constant $M_i^{(1)}$. Repeating this process $n - 1$ times, we are led to a Cauchy problem of the form

$$U_{zz^*}^{(n-1)} = F(z, z^*, A_1^{(n-1)}(U^{(n-1)}),..., A_N^{(n-1)}(U^{(n-1)})), \tag{2.12}$$

$$\begin{aligned} U^{(n-1)}(z, z^*) &= \varphi_0^{(n-1)}(z); \qquad z = z^*, \\ i\left(\frac{\partial U^{(n-1)}}{\partial z} - \frac{\partial U^{(n-1)}}{\partial z^*}\right) &= \varphi_1^{(n-1)}(z); \qquad z = z^*, \end{aligned} \tag{2.13}$$

where

$$U^{(n-1)}(z, z^*) = \frac{\partial^2 U^{(n-2)}}{\partial z\, \partial z^*} \tag{2.14}$$

and $A_i^{(n-1)}$, $i = 1,..., N$, are integral operators on HB into HB which satisfy the condition

$$\begin{aligned} &\| A_i^{(n-1)}(U_1^{(n-1)}) - A_i^{(n-1)}(U_2^{(n-1)})\| \\ &\leqslant M_i^{(n-1)} \Big\{\| U_1^{(n-1)} - U_2^{(n-1)} \| + \left\| \frac{\partial U_1^{(n-1)}}{\partial z} - \frac{\partial U_2^{(n-1)}}{\partial z} \right\| \\ &\quad + \left\| \frac{\partial U_1^{(n-1)}}{\partial z^*} - \frac{\partial U_2^{(n-1)}}{\partial z^*} \right\| \Big\} \end{aligned} \tag{2.15}$$

for some positive constant $M_i^{(n-1)}$.

We note that the operators $A_i^{(k)}$, $i = 1,..., N$, $k = 1,..., n - 1$, all turn out to be integral operators satisfying a condition such as (2.11), due to the fact that we restricted ourselves to a rather special class of semilinear equations. For equations not of the form (1.4) (e.g., Eq. (2.3)), the operator $A_i^{(k)}$ would fail to satisfy such conditions for $k > k_0$, where k_0 is some integer less than $n-1$.

We now proceed to use the contraction mapping principle to find a solution of Eqs. (2.12), (2.13). By hypothesis, F is holomorphic in a compact subset of the space of $N+2$ complex variables and, hence, from Schwarz's lemma for functions of several complex variables [3, p. 38], a Lipschitz condition holds there with respect to the last N arguments, i.e.,

$$\begin{aligned} &| F(z, z^*, \xi_1, \ldots, \xi_N) - F(z, z^*, \xi_1^0, \ldots, \xi_N^0)| \\ &\qquad \leqslant C_0\{| \xi_1 - \xi_1^0 | + \cdots + | \xi_N - \xi_N^0 |\}, \end{aligned} \tag{2.16}$$

where C_0 is a positive constant. Hence, by (2.15) and (2.16), there exists a positive constant C_1 such that

$$\begin{aligned} &\| F(z, z^*, A_1^{(n-1)}(U_1^{(n-1)}), \ldots, A_N^{(n-1)}(U_1^{(n-1)})) \\ &\qquad - F(z, z^*, A_1^{(n-1)}(U_2^{(n-1)}), \ldots, A_N^{(n-1)}(U_2^{(n-1)}))\| \\ &\leqslant C_1 \left\{\| U_1^{(n-1)} - U_2^{(n-1)} \| + \left\| \frac{\partial U_1^{(n-1)}}{\partial z} - \frac{\partial U_2^{(n-1)}}{\partial z} \right\| \right. \\ &\qquad \left. + \left\| \frac{\partial U_1^{(n-1)}}{\partial z^*} - \frac{\partial U^{(n-1)}}{\partial z^*} \right\| \right\}. \end{aligned} \tag{2.17}$$

It should be noted that $A_i^{(n-1)}$ are in fact integral operators on $U_1^{(n-1)}$ and its derivatives with respect to z and z^*, i.e. $A_i^{(n-1)}(U_1^{(n-1)}) \equiv A_i^{(n-1)}(U_1^{(n-1)}, \partial U_1^{(n-1)}/\partial z, \partial U_1^{(n-1)}/\partial z^*)$.

Now define the operators B_i, $i = 1, 2, 3$, by

$$s(z, z^*) = U_{zz^*}^{(n-1)}(z, z^*), \tag{2.18}$$

$$\begin{aligned} B_1(s) \equiv U^{(n-1)}(z, z^*) = \int_0^z \int_0^{z^*} s(\xi, \xi^*)\, d\xi^*\, d\xi + \int_0^z \gamma(\xi)\, d\xi \\ + \int_0^{z^*} \psi(\xi^*)\, d\xi^* + \varphi_0^{(n-1)}(0), \end{aligned} \tag{2.19}$$

$$B_2(s) \equiv U_z^{(n-1)}(z, z^*) = \int_0^{z^*} s(z, \xi^*)\, d\xi^* + \gamma(z), \tag{2.20}$$

$$B_3(s) \equiv U_{z^*}^{(n-1)}(z, z^*) = \int_0^z s(\xi, z^*)\, d\xi + \psi(z^*), \tag{2.21}$$

where [1]

$$\gamma(z) = \frac{1}{2}\left[\frac{d\varphi_0^{(n-1)}(z)}{dz} - i\varphi_1^{(n-1)}(z)\right] - \int_0^z s(z, \xi^*)\, d\xi^*, \tag{2.22}$$

$$\psi(z) = \frac{1}{2}\left[\frac{d\varphi_0^{(n-1)}(z)}{dz} + i\varphi_1^{(n-1)}(z)\right] - \int_0^z s(\xi, z)\, d\xi. \tag{2.23}$$

Finding a solution to the Cauchy problem (2.12), (2.13) is now equivalent to finding a fixed point in the Banach space HB of the operator $T : HB \to HB$, defined by

$$Ts = F(z, z^*, A_1^{(n-1)}(B_1(s)),\ldots, A_N^{(n-1)}(B_1(s))). \tag{2.24}$$

For a given a, $0 \leqslant a < 1$, and ρ sufficiently small, it is easily seen from Eqs. (2.18)–(2.23) that

$$\| B_i s_1 - B_i s_2 \| \leqslant \frac{a}{3C_1} \| s_1 - s_2 \| ; \qquad i = 1, 2, 3. \tag{2.25}$$

Hence, from Eqs. (2.17) and (2.24), we have

$$\| Ts_1 - Ts_2 \| \leqslant a \| s_1 - s_2 \| \tag{2.26}$$

and

$$\| Ts \| \leqslant a \| s \| + \| To \| \leqslant \| s \| + (1 - a) M_0 , \tag{2.27}$$

for some positive constant M_0. Hence, if $\| s \| < M_0$, then $\| Ts \| < M_0$, i.e., T is a contraction mapping of a closed ball of HB into itself. Hence T has a (unique) fixed point $s(z, z^*)$ and, therefore, Eqs. (2.19), (2.22), (2.23) give the solution $U^{(n-1)}(z, z^*)$ to (2.12), (2.13). Now, refering back to Eqs. (2.7) and (2.14), we see that

$$U^{(k-1)}(z, z^*) = \int_{z^*}^{z} \left\{ \int_{\xi}^{z^*} U^{(k)}(\xi, \xi^*)\, d\xi^* + \frac{1}{2}\left[\frac{d\varphi_0^{(k)}(\xi)}{d\xi} - i\varphi_1^{(k)}(\xi)\right]\right\} d\xi + \varphi_0^{(k)}(z). \tag{2.28}$$

Hence, from a knowledge of $U^{(n-1)}(z, z^*)$, we immediately obtain the solution $U(z, z^*)$ to our original Cauchy problem (2.1), (2.2), by a series of quadratures.

THEOREM. *There exists a constructive procedure, suitable for analytic approximations, for solving the Cauchy problem* (1.4), (1.5). *Such a procedure is given explicitly by* (2.1)–(2.28).

It is important to note that the unstable dependence of the solution of the elliptic Eq. (1.4) on the (real) Cauchy data (1.5) appears exclusively in the step where this data is extended to complex values of the independent variable x. When this can be done in an elementary way, for example, by direct substitution via the transformation (1.2), no instabilities will occur when one uses the contraction mapping operator T to obtain approximations to the desired solution.

We finally note that if equation (1.4) is linear and one uses exponential majorization (c.f. [1], [3]), then the above techniques yield *global* solutions to Cauchy's problem. In particular if the norm (2.10) is taken over $\mathfrak{S} \times \mathfrak{S}^*$ instead of $\Delta\rho \times \Delta\rho^*$, we obtain an extension of Henrici's theorem ([4], p. 196) to higher order elliptic equations.

References

1. D. Colton, Cauchy's problem for almost linear elliptic equations in two independent variables, *J. Approximation Theory* **3** (1970), 66–71.
2. P. R. Garabedian, "Partial Differential Equations," Wiley, New York, 1964.
3. R. P. Gilbert, "Function Theoretic Methods in Partial Differential Equations," Academic Press, New York, 1969.
4. P. Henrici, A survey of I. N. Vekua's theory of elliptic partial differential equations with analytic coefficients, *Z. Angew. Math. Phys.* **8** (1957), 169–203.
5. I. N. Vekua, "New Methods for Solving Elliptic Equations," Wiley, New York, 1967.